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1. Introduction

Statistical Mechanics is a mature discipline. As is the case in Quantum Mechanics,^(1,2) physicists know how to calculate any quantity of interest, and no one doubts it will be in complete agreement with experiment. However, some physicists feel uneasy with the interpretation of the rules of the game, because it appears to be in conflict with the nearest and dearest part of their intuition. This Essay is concerned with the meaning of entropy; as evidence of the efforts to reach a harmonious position in this issue, let me recall Refs. 3-4.

The second law of thermodynamics rose as an empirical fact and asserted that the useful energy available from an isolated system will always decrease (and eventually approach zero). This degradative behaviour is predicted not only for simple processes (such as diffusion, heat conduction or explosions), but for every possible process, even if methodic actions are taken by intelligent beings in the isolated system, no matter how ingenious they might be. The only exception to this degradative behaviour are the so called “reversible processes”, in which useful energy is kept unused. In Section 2 I introduce “the rebel”, and attempt to analyze the reasons he has for not liking this form of the second law.

With Lord Kelvin and Clausius, the second law became more formal. Kelvin’s formulation states that if a closed system is at uniform temperature, then it is impossible to transform part of its internal energy into mechanical energy. Using the second law Clausius proved that, for T being the absolute temperature and $d'Q$ the heat entering the system in an infinitesimal reversible process, $1/T$ is an integrating factor for $d'Q$. This result translates the second law into the following assertions.

a. If a system is in equilibrium, there exists a function of state S (called the

entropy), such that

$$dS = d'Q/T . \tag{1}$$

b. If a process occurs in an isolated system, the final entropy will not be less than the initial entropy.

The next stage of evolution of the second law came in with Boltzmann's *H*-theorem.⁽⁵⁾ He built a function and showed that, under apparently innocent assumptions, it behaves as the entropy. Boltzmann's theorem changed the status of the second law from an experimental fact to a consequence of dynamics. It also provided the previously abstract function *S* with a meaning in microscopic terms (*S* could then be interpreted as a measure of disorder). The most controversial feature of the *H*-theorem is the apparition of time-asymmetry in the behaviour of a system which evolves under time-symmetric laws. As a hopefully representative short list of the never ending debate raised by the *H*-theorem, I have picked Refs. 6-10. Boltzmann's theorem will be examined in Section 3.

Part of the controversial nature of the second law is probably due to the usage of the same words by different authors, though they have different concepts in mind. In order to be on the safe side, I will stick to the definitions in Section 3. No claim is made that the definitions given here are necessarily identical with those used by the majority of the scientific community. Hopefully, the adoption of a non-conventional view will shed light and stimulate thought to a greater degree than it will cause confusion. Section 3 provides the foundations for the discussion in Sections 5 and 6.

In Section 4 the rebel will be given the chance to beat the second law. However, his efforts fail. In the remainder of this article, the rebel will have to learn how to live with this law.

With the advent of information theory,⁽¹¹⁻¹⁵⁾ it became natural to identify entropy with lack of information. This is done in Section 5, which is a corridor towards the closing section. The connection between information and entropy

was hazy until Jaynes^(16–19) and followers^(20–22) cast it into precise terms.

Section 6 contains what could be called the output of this article. The suggested conclusions are:

- a.* Disorder and lack of information are not the same thing.
- b.* The notion of “encoded order” is introduced.
- c.* Quantities which are traditionally considered to be completely objective, such as temperature, do depend on the awareness of the thermometer designer about the state of the system.
- d.* The correct formulation of the second law should take (*c*) into account.

It is my conviction that the interpretation given here was already implicit in the thinking of many physicists, such as J.C. Maxwell,⁽²³⁾ A. Einstein, R.C. Tolman and E.T. Jaynes. Perhaps the feature which distinguishes this article from all others is its attitude towards “the rebel”. His obstinacy is not tagged as blindness to facts, but is regarded as an expression of honest, courageous and creative thinking.⁽²⁴⁾

The mathematical definition of “randomness” introduced by Solomonoff,⁽²⁵⁾ Kolmogorov⁽²⁶⁾ and Chaitin⁽²⁷⁾ has led several authors^(28,29) to associate a larger entropy to *microstates* which are “less simple”. This point will be addressed in Section 6.

A more popular version of this article will appear elsewhere.⁽³⁰⁾

For the sake of completeness, I conclude this introduction by mentioning related subjects which will not be treated here.

- a.* While Schroedinger’s equation is invariant under time-reversal, the “collapse” of the wave function is not. This led von Neumann⁽³¹⁾ to regard quantum measurements as the ultimate reason for irreversibility. This aspect is given only indirect consideration in Section 3.
- b.* In principle, the second law applies to “isolated systems”. If we define them as systems which do not interact with anything else, then the only truly isolated system would be the entire universe. But if isolation implies constant energy and volume, then the universe is an illegal example, since

it is expanding. In addition, the universe is not a typical thermodynamic system, since, as a consequence of the long range of gravitation, the energy and the entropy are not extensive quantities. Some references which consider the cosmologic aspects of thermodynamics are 9 and 32-34. The connection between the ideas in this article and the expansion of the universe is discussed in Ref. 30.

2. The Rebel's Standpoint

The rebel does not respect the second law, which is an inequality rather than an equality. Since an inequality is more likely to be fulfilled by chance - as compared to an equality, it is experimentally less convincing.

The rebel has great self esteem, and does not accept the extrapolation that mankind will never be able to contrive in the future what couldn't be contrived in the past. When the rebel encounters a "proof" for the second law, his/her conviction is that the only thing proven is the lack of imagination of the prover.¹ The rebel never surrenders without a fight;⁽²⁴⁾ being a lone fighter only enhances his/her sense of mission.

In spite of the long time elapsed since the universe began to evolve, the rebel looks around and discovers an amazing amount of order. Moreover, he/she is aware of the existence of "historically irreversible" processes, such as the carving of a bed by a river, a "favourable mutation" in Darwinian theory, or a discovery in a communicating society. After being triggered by these processes, the produced changes build up and increase the amount of order perceived by their observer. All these phenomena seem to be opposed to the trend predicted by the second law.

3. Nomenclature and Boltzmann's Theorem

3.1 The statistical state and its evolution

The definitions used here follow the line of thinking of Ref. 21. I shall use a quantum description of the system, not only because it is the true

representation of nature, but also because it is more “natural”: it provides a natural way of counting states and assigning them prior probabilities and a natural framework for a probabilistic evolution. Besides, it is more appealing to use sums rather than integrals. Nevertheless, I shall have in mind a macroscopic system, so that its behaviour in classical terms will also be meaningful.

Let $\psi_1, \psi_2, \dots, \psi_N$ be a basis of the Hilbert space of the possible states of the system. (If the system is isolated and has finite volume, energy and degrees of freedom, then N is finite too.) From this paragraph on, whenever I use the word “state” (or “microstate”) without adding any “family name”, I mean quantum state of the system. Since the system is isolated, some quantities, such as the energy or the volume, remain constant. Only states with the appropriate value for those quantities (possibly within some tolerance) are possible. The choice of the basis in the Hilbert space is arbitrary, and we usually select the basis which provides the simplest description to the best of our knowledge.

By performing measurements, we can determine quantities such as the mass of the upper half of the system, the magnetization of the southern third of the system or the average pressure against its western wall. Such quantities are called “macroscopic coordinates” of the system. A set of macroscopic coordinates is called “complete” (within a given set of constraints) if preparation of their initial values determines reproducibly their evolution in time (allowing for the possible exception of an initial transient time). Note that for the same given physical system there may be several complete sets of macroscopic coordinates (some of them more detailed than others), depending on which quantities we are able or willing to measure. A complete set of values for the macroscopic coordinates is called the “thermodynamic state” of the system.

Generally, there will be a huge number of quantum states (i.e. microstates) that will fit the values of all the macroscopic coordinates in the

complete set. Therefore, knowledge of the thermodynamic state of a system does not determine its quantum state. As a consequence, the best we can do is to infer the probabilities² P_i for the system being in the state ψ_i ($i = 1, 2, \dots, N$). In the present discussion we disregard the possibility of the system being in a state $\sum c_i \psi_i$, with the coefficients c_i having well defined phases. This policy is legitimate provided that the macroscopic coordinates are insensitive to these phases, either because they are diagonal in the basis $\{\psi_i\}$ or because the contribution of the phases averages to zero. The set of probabilities P_1, P_2, \dots, P_N is called here the “statistical state” of the system. (The statistical state is usually called the “ensemble” and the present policy about phases, the “random phases assumption”.)

Is the statistical state “objective” or “subjective”? The precise answer is that, for a given set of macroscopic coordinates and a given basis of the Hilbert space (which are chosen), the P_i ’s are *functions*^(16,20) of the values obtained for the macroscopic coordinates (which are measured). The sorting of this situation is a matter of semantics.

Let us now study how the statistical state evolves. If at time t_o the system were in a given state ψ_i , then at time $t_o + \Delta t$ it would be in a linear combination of many of the states ψ_j . Since for complicated systems we cannot follow quantum phases, such linear combinations look to us as mixtures, and our best description limits itself to assigning a probability W_{ij} for the passage from any initial i to any final j . The values W_{ij} depend on the dynamical details of the system and on the elapsed time Δt . It follows that if the statistical state at time t_o was P_1, P_2, \dots, P_N , then the statistical state P'_1, P'_2, \dots, P'_N at time $t_o + \Delta t$ will be given by

$$P'_j = \sum_i W_{ij} P_i, \quad (2)$$

where I have assumed Markovian evolution.⁽³⁵⁾ Clearly,

$$\sum_j W_{ij} = 1, \quad (3)$$

since the system has to be in some state j at time $t_o + \Delta t$. In addition the linearity of Schroedinger's equation implies that every ψ_i will evolve into $\mathcal{U}\psi_i$, where \mathcal{U} is a linear operator. Since the norms of wave functions do not change, \mathcal{U} must be unitary. Therefore, taking $W_{ij} = |(\psi_j|\mathcal{U}\psi_i)|^2$, we also obtain

$$\sum_i W_{ij} = 1 . \quad (4)$$

The set $\{P'_j\}$ in (2) is the most unbiased statistical state at time $t_o + \Delta t$ which can be predicted from knowledge of the thermodynamic state at time t_o . This is not necessarily the same as the statistical state that would be inferred if the complete set of macroscopic coordinates were actually measured at $t_o + \Delta t$. We could predict $\{P'_j\}$ with certainty only if we could predict with certainty the thermodynamic state. It is tempting to argue that for a macroscopic system and for Δt negligible compared to Poincaré's recursion time, the variances in $\{P'_j\}$ will be very small and prediction (2) will be highly reliable. However, by definition, this argument is unnecessary: if the macroscopic coordinates (and therefore $\{P'_j\}$) after an experimentally relevant time Δt were not determined uniquely by their initial values at t_o , then they would not constitute a complete set.³

The evolution rule (2) will be refined in Subsection 3.3.

3.2 The entropy and its evolution

Since it seems that no definition of entropy is universally accepted, except for Eq. (1), let me specify the choice of this article. The entropy will be temporarily given by

$$S = -kH \quad (5a)$$

with k Boltzmann's constant and

$$H = \sum_i P_i \ln P_i . \quad (5b)$$

This definition was used by Klein.⁽³⁶⁾ It will be revised in the next subsection. H can be written in the form

$$H = \text{Tr}(\rho \ln \rho), \quad (5c)$$

where ρ the reduced density matrix. The reduced density matrix is obtained from the density matrix by substituting the non-diagonal elements by zeroes. An ugly feature of (5c) is that H depends on the particular basis $\psi_1, \psi_2, \dots, \psi_N$ which we choose. It should become apparent below that this ugliness is not a matter of mathematical clumsiness, but is an essential feature of the H -function.

Definition (5) may be considered legal only if it reduces to definition (1) when the latter is applicable, namely, infinitesimally close to a situation in which the temperature is an equilibrium property and energies can be attributed to the possible states of the system. This implies the canonical distribution

$$\ln P_i = -E_i/kT - \ln Z, \quad (6)$$

where E_i is the energy of the system if it is at state ψ_i and the partition function Z is independent of i . Identifying the internal energy with the expectation of E , and the work on the system as the expectation of the change in E , we obtain

$$d'Q = d\left(\sum_i P_i E_i\right) - \sum_i P_i dE_i = \sum_i E_i dP_i. \quad (7)$$

On the other hand, using (5), (6) and noting that $\sum_i P_i$ does not change

$$TdS = -kT \sum_i (1 + \ln P_i) dP_i = \sum_i E_i dP_i, \quad (8)$$

in agreement with (1).

Definition (5) is not only a generalization of (1); it also has a meaning: H is the unique reasonable additive measure of the information^(11,12,21) about the state of the system.

Let us now show that the entropy defined by (5) does not decrease with time. The following lines are a translation of the standard proof^(8,37) to present notation. The evolution rule (2) gives

$$H(t_o + \Delta t) = \sum_j P'_j \ln P'_j = \sum_{ij} W_{ij} P_i \ln P'_j, \quad (9)$$

whereas $H(t_o)$ can be written in the form

$$H(t_o) = \sum_i \left(\sum_j W_{ij} \right) P_i \ln P_i = \sum_{ij} W_{ij} P_i \ln P_i. \quad (10)$$

Therefore, since for any pair (x, y) of positive numbers $x(\ln x - \ln y) \geq x - y$, it follows from here and (4) that

$$\begin{aligned} H(t_o) - H(t_o + \Delta t) &= \sum_{ij} W_{ij} P_i (\ln P_i - \ln P'_j) \geq \\ \sum_{ij} W_{ij} P_i - \sum_{ij} W_{ij} P'_j &= \sum_j P'_j - \sum_j P'_j = 0, \end{aligned} \quad (11)$$

as required by the second law.

Since Boltzmann was the first to build a function of the statistical state of the system, such that it decreases as equilibrium is being approached, inequality (11) will be called Boltzmann's theorem. However, let me emphasize that definition (5b) is not equivalent to Boltzmann's H . A nonessential difference is that Boltzmann's H was classical. An essential difference is that, whereas the index i in (5b) refers to the state of the entire system, Boltzmann's H was expressed in terms of the single-particle distribution function, and could not take correlations into account. As made clear by Jaynes^(38,39), Boltzmann's H does not necessarily fulfill inequality (11) for a system of interacting particles.

3.3 Validity and significance of Boltzmann's theorem

Let us first recognize that the range of application of the second law has been greatly enlarged, from situations in which the system is in equilibrium

both at t_o and at $t_o + \Delta t$ and a reversible path connects between them, to any situation in which a given complete set of macroscopic coordinates has well defined values at t_o and, as a consequence, $\{P_i\}$ and $\{P'_i\}$ are well defined, too. Nevertheless, let us note that definition (5) is only a possible generalization of (1) and, in the absence of additional criteria, other generalizations may be possible, too.

Next, let us note that an essential ingredient in the proof of (11) is the assumption of a probabilistic rather than a deterministic evolution of the quantum state. If we knew that every state ψ_i evolves during time Δt into the state $\psi_{j(i)}$, then W_{ij} would vanish for $j \neq j(i)$, equation (2) would become $P'_{j(i)} = P_i$, and $W_{ij}(\ln P_i - \ln P'_j)$ would become zero. The absence of determinism may be traced to the fact that measurement of the thermodynamic state is insensitive to phases; therefore, the thermodynamic evolution can only be described by averaging out the actual phases, and retaining only the probabilities for the possible outcomes.

Indeed, had we taken ρ as the full density matrix in definition (5c), H would remain constant in time due to the unitarity of the operator \mathcal{U} . This is most disturbing, since, with standard modifications of (6)-(8), the expressions for $d'Q$ and for TdS would become

$$d'Q = \text{Tr}(\mathcal{H}d\rho) \quad (12)$$

(where \mathcal{H} is the Hamiltonian) and (5) would still be a generalization of (1). Let us now consider the case that the system is in equilibrium at $t_o + \Delta t$. How can two expressions for the entropy both reduce to (1) at $t_o + \Delta t$ and nevertheless have different values at $t_o + \Delta t$? The answer is that expressions (7) and (12) for $d'Q$ are different.

If so, we cannot decide which is the correct generalization for (1) before we decide on the correct expression for $d'Q$. The old specification prescribed by thermodynamics is that heat is the amount of energy transfer which cannot be accounted for as work, whereas work is the change in mechanical energy

involved in the changes of the macroscopic coordinates. Therefore, an expression for the work which involves changes in quantum phases, while none of the macroscopic coordinates is sensitive to them, is not thermodynamically acceptable. Eq. (12) must thus be rejected.

After learning this lesson, we should revise our definition of entropy. For any realistic situation, the macroscopic coordinates will not only be insensitive to the phases. There will be classes of states G_1, G_2, \dots, G_M , $M \ll N$, such that if ψ_i and ψ_j are both in the same class G_ν , then they will be experimentally indistinguishable by measurement of any of the macroscopic coordinates. Therefore, in the thermodynamic definition (7) of heat we should replace the separate values P_i for each state by common values within every class. At measurement time t_o , due to the complete inability to distinguish among the states in G_ν , all the P_i 's for $\psi_i \in G_\nu$ will be equal anyway. However, after time Δt , the evolution rule (2) predicts in general different values P'_j for states j in the same class. These differences are thermodynamically irrelevant. The only thermodynamically relevant probabilities are those of the entire classes, which equal the respective sums of the P'_j 's pertaining to them. If the system behaves reproducibly between t_o and $t_o + \Delta t$, measurement of the macroscopic coordinates at $t_o + \Delta t$ should indeed give results which are compatible with the predicted probabilities for the classes. However, the least biased probabilities to be associated with every microstate would not be the $\{P'_j\}$, but rather the probabilities $\{\overline{P'_j}\}$ which, while being compatible with the probabilities of the classes, provide the smallest possible information about the actual microstate of the system. Since the smallest information is available when all the probabilities within any given class are equal,

$$\overline{P'_j} = \text{average of the } \{P'_j\} \text{ in the class of } j. \quad (13)$$

This averaging of the evolved statistical state predicted by (2) is usually called “coarse graining”.⁽⁸⁾ According to the discussion here, a better name

would be “thermodynamic relevance”.⁽²²⁾

It follows from the previous paragraph that the relevant entropy at $t_o + \Delta t$ is not given by the $H(t_o + \Delta t)$ which is obtained directly from (2). Rather, we should use the value $\bar{H}(t_o + \Delta t)$ which is obtained by substituting $\{P'_j\} \rightarrow \{\bar{P}'_j\}$ into the expression (5b) for H . From (11) and from the construction of (13)

$$\bar{H}(t_o) = H(t_o) \geq H(t_o + \Delta t) \geq \bar{H}(t_o + \Delta t) , \quad (14)$$

so that the entropy increases *a fortiori*. The meaning of inequality (14) is that, if the thermodynamic state of an isolated system is known at time t_o , it will evolve during the time interval Δt in a way that implies losing information about the actual microstate. As a consequence, we will lose part of our capability of utilizing the energy of the system.

If the states ψ_i are good approximations to the eigenstates of the Hamiltonian, then $W_{ij} = W_{ji}$, due to Fermi’s golden rule. This condition is stronger than Eq. (4) and is not always valid.⁽⁴⁰⁾ This stronger condition was assumed in Pauli’s master equation⁽⁴¹⁾ to prove a stronger inequality than (14), namely, $dH/dt \leq 0$. For a hierarchical analysis of H -theorems, see Ref. 42.

I shall close this subsection with the answers to two perennial questions:

a. Is the entropy objective or subjective?

Since the entropy is a function of the statistical state, both are objective to the same extent. Thus, the answer was already given in Subsection 3.1.

b. If we retrodict the statistical state at $t_o - \Delta t$ from the statistical state at t_o , Eqs. (2) and (4) remain valid. From them, we can use the same arguments that lead to (14) to prove that $\bar{H}(t_o - \Delta t) \leq \bar{H}(t_o)$. Actually, if we invoke the invariance of Schroedinger’s equation under time reversal, we reach a stronger conclusion: the statistical state at $t_o - \Delta t$ is the same as at $t_o + \Delta t$. Is it true that the entropy decreases during the time interval $(t_o - \Delta t, t_o)$?

If it were true that the only information we have about the state of the system comes from the measurements performed at t_o , and we had no idea

about its state in the past, then the answer to this question would be “yes”. This is not the usual experimental situation. In order to have a meaningful experiment, we have to prepare its initial thermodynamic state. This is done either by waiting for equilibrium and then removing some constraint, or by following the thermodynamic evolution until a definite regularity is detected. This implies knowledge about the system during some lapse of time before t_o . The asymmetry between past and future comes from the fact that we usually gather data about the past of the system, but we cannot gather data about its future. This question is treated in more detail in Refs. 30, 34, 43 and 44.

3.4 Non-isolated systems

Thermodynamics does not forbid the decrease of entropy for a system which is not isolated. Where in the proof of Eq. (14) have we required isolation of the system?

First, I have assumed that the Hilbert space remains unchanged ($\{\psi_i\}$ is still a basis at $t_o + \Delta t$). This would not be the case if the volume, or the shape, or the energy, or the Hamiltonian, or the degrees of freedom of the system were to change. Note, however, that definition (5) permits an obvious generalization: for a “microscopically adiabatic” process, such that the Hilbert space or the energy levels change, but no jumps among the states $\{i\}$ are induced, the statistical state and the entropy do not undergo any immediate change.

Second, I have assumed that, in principle, pure states remain pure. Namely, the state of the system does not become entangled with the state of some other system. Entanglement would not affect the evolution rule (2), but it would affect its interpretation.

Finally, I have used Eq. (4). In order to appreciate Eq. (4), let us consider a system which is not isolated and see that Eq. (4) turns into something else.

Let us examine the case of a system in thermal contact with its envi-

ronment. By this we mean a system which has to be considered together with a companion system, called the “environment” in order to constitute an isolated system. Let their total energy be E_T (possible unknown). The coupling between them is sufficiently weak, so that we can associate separate states to the system and to the environment, with respective energies E and $E_T - E$. On the other hand, this coupling is sufficiently strong to bring about changes in E . For every value of E , let there be $\omega(E)$ possible states for the system, which we shall denote by i , $1 \leq i \leq \omega(E)$. Similarly, the state of the environment will be denoted by I , $1 \leq I \leq \Omega(E_T - E)$. We shall attempt to keep track of the state of the system (in a statistical sense); on the other hand, we shall have no information either about the state of the environment or about any possible correlation between i and I .

The state of the entire system will be determined by the three indices E, i and I , but we keep track only of the first two. The probability for a state E, i of the system will be denoted by $P_{Ei} = P_E P_i(E)$ where $P_i(E)$ is the probability of the state E, i , given the fact that its energy is E . Due to our complete ignorance about the state of the environment, the probability $P_I(E)$ for a state I when E is known is simply

$$P_I(E) = 1/\Omega(E_T - E) . \quad (15)$$

From the absence of correlation it follows for the state of the entire system

$$P_{EiI} = P_{Ei}/\Omega(E_T - E) . \quad (16)$$

Let us now consider the probability $W_{Ei;E'i'}$ for passage of the system from E, i to E', i' . We wish to express $W_{Ei;E'i'}$ in terms of $W_{EiI;E'i'I'}$ - the probability for passage of the entire system from E, i, I to E', i', I' . Since the environment might initially be in any state I , and must finally be in some state I' , we have

$$W_{Ei;E'i'} = \sum_{II'} P_I(E) W_{EiI;E'i'I'} = \frac{1}{\Omega(E_T - E)} \sum_{II'} W_{EiI;E'i'I'} . \quad (17)$$

Since the entire system is isolated, we can apply Eq. (4) to $W_{EiI;E'i'I'}$. Using (17), it follows that

$$\sum_{Ei} W_{Ei;E'i'} \Omega(E_T - E) = \sum_{EiII'} W_{EiI;E'i'I'} = \sum_{I'} 1 = \Omega(E_T - E') . \quad (18)$$

This is the relationship we were looking for: if a system is in thermal contact with its environment, Eq. (4) has to be replaced by Eq. (18). A simple meaning of Eq. (18) is that a statistical state $P_{Ei} \propto \Omega(E_T - E)$ is stationary.

Let us now develop a version of Boltzmann's theorem, adapted to the present system. Using (5b), (16), and summing over I , we obtain the H -function for the system + environment:

$$H_T = \sum_{Ei} P_{Ei} \ln \frac{P_{Ei}}{\Omega(E_T - E)} . \quad (19)$$

Although H_T is the value of H for the system + environment, it is expressed in (19) in terms of the statistical state of the system only. This result reflects the fact that H measures information and we have no information about the state of the environment. Since the entire system is isolated, we know already that $H_T(t_o + \Delta t) \leq H_T(t_o)$. It is an instructive exercise to show the decrease in H_T by following the evolution of $\{P_{Ei}\}$. This is done by the same steps which lead to inequality (11), but condition (4) has to be replaced by (18).

Eq. (18) has the unsatisfactory feature that it involves the structure of the environment (the function Ω). We can get rid of it when the environment is a “reservoir”, whose temperature is always T . From (5) and (15),

$$\Omega(E_T - E) = e^{S_{\text{res}}/k} , \quad (20)$$

where S_{res} is the entropy of the reservoir when the system has energy E . The temperature being constant and the work being zero, it follows from definition (1) that

$$S_{\text{res}} = S_o - E/T , \quad (21)$$

where S_o is the entropy of the reservoir when the energy of the system vanishes. Introducing (20) and (21), condition (18) reduces to

$$\sum_{Ei} W_{Ei;E'i'} e^{-E/kT} = e^{-E'/kT} . \quad (22)$$

When the environment is a reservoir, H_T has a simple thermodynamic meaning. Using (20) and (21), and dropping an additive constant, H_T reduces to $\langle E \rangle / kT - S/k$, where S is the entropy of the system and $\langle E \rangle$ is the expectation for its energy. Therefore, kTH_T is the Helmholtz free energy, which is the appropriate thermodynamic potential for a system kept at constant temperature while no work is done on it.

4. Attempts to Beat the Second Law

The previous section proves that the entropy of an isolated system will not decrease. But you never know; perhaps the proof, and the massive experimental evidence too, overlook some critical ingredient which might do the job. Therefore, let us be guided by the rebel's ingenuity and devote one section to examine situations where this kind of ingredient appears to be present.

4.1 Maxwellian Demons

The best known challenge to the second law was posed by J.C. Maxwell in 1871. He considered a volume of gas, initially at uniform temperature, divided into two chambers by a partition with a door. Whenever a molecule attempts to go through, a “demon” decides whether to open or to close the door, depending on the speed of the molecule. In this way, the demon can manage to build a high concentration of fast molecules in one chamber and slow molecules in the other. This would mean a decrease in the entropy of the gas. To be sure, there is nothing sacred about the speed of the molecules. As long as the demon can detect some feature of a molecule and behave selectively depending on whether it attempts to cross from chamber one to

two or from two to one, a difference between the chambers will build up and this difference permits the extraction of useful energy from the gas.

In order to pose a real challenge to the second law, the mysterious demon of the previous paragraph has to be replaced by a conceivable physical device. This physical “demon” should be some sort of unidirectional valve as encountered, for instance, in most inflatable objects. Two idealized candidates for this job are shown in figures 1 and 2.

Figure 1 shows a volume of gas, initially at uniform pressure, divided into two chambers by a partition with a door which can open to one side only (say, to the left). A molecule (such as M) coming from the right will open the door and cross to the other chamber; the inverse process (M') is inhibited. As a consequence, molecules should accumulate at the left chamber and the pressure difference could be used to perform work. This demon, and his exorcism, were proposed by Smoluchowski.⁽⁴⁵⁾ Situations which are apparently equivalent to that of Fig. 1, though they are less transparent, are studied in Refs. 46 and 47.

Figure 2 shows a resistor R held at a positive absolute temperature. Due to thermal motion of the electrons, a random voltage builds up between the ends of R (Johnson noise). This voltage cannot be used, because it changes sign with an unpredictable rhythm. However, if we add a diode D in series, we should obtain a rectified current I which always circulates counterclockwise, and could be used to run motor M . This possibility has been considered by several authors.⁽⁴⁸⁾

Devices of this sort do indeed work... provided that the physical “demon” is sufficiently cold. There is no particular novelty in these devices: they are just heat engines whose heat intake is at the gas (or at the resistor R), their heat sink is at the “demon”, and the difference in heat flows is used to perform work. They should not violate the second law more than any other existing heat engine.

The flaw in these devices (from the rebel’s point of view) is that the

“demon” heats up during operation. When the “demon” is hot, it suffers from thermal random movements which cause “him” to perform an imperfect task. For instance, the door in Fig. 1 will begin to open and close by itself, even though no molecule is hitting it. It may so happen that the door is open as shown in the figure, the spring is closing it, and a molecule is tracing a trajectory inverse to that of M . Then this molecule would be kicked by the door into the right chamber (that is, into “the wrong” chamber), in spite of the fact that it had no plans of going there.

Similarly, for positive absolute temperatures, the diode D not only acts as a rectifier, but also as a d.c. bias in “the wrong” direction. We may therefore expect, as it happens for every heat engine, that the device will stop working at the moment that the temperature of the “demon” becomes equal to the temperature of the heat source. If the demon’s temperature becomes larger than that of the “heat source”, then the device should operate in “the wrong” direction.

The rebel could rightly argue that the previous paragraph is a plausibility argument rather than a proof. The imperfections of the demon should diminish his performance, but, before we see an explicit calculation, why should we believe that the demon’s action will precisely cancel? Moreover, when calculating the work performed by the demon, we should avoid using the rules of statistical mechanics, or even the concept of temperature, which already have the second law of thermodynamics built into them; only Newton’s laws (or Schroedinger’s equation) and mathematics are acceptable. Calculations of this sort have been recently performed:⁽⁴⁹⁾ simulations show indeed that while the door in Fig. 1 is able to act as a unidirectional valve when large pressure differences between the chambers are present, it does not rectify fluctuations unless it is artificially cooled.

Smoluchowski concluded that “there is no automatic, permanently effective perpetual motion machine, in spite of molecular fluctuations, but such a device might perhaps function regularly if it were operated by intelligent

beings.” “Intelligent beings” differ from “automatic Maxwellian demons” in that they have to take measurements and make decisions. These subjects will be dealt with in Section 5.

For a review of the literature on Maxwellian demons, see Refs. 50 and 51.

4.2 Geometry-based guides

If you can’t beat them – join them. In this subsection we consider situations in which it is not attempted to attain rectification by granting a preferred direction to the molecules, but, on the contrary, by removing some preferred direction they already had. The role of the demon, who is merely the walls of the container, is just to make the impinging molecules “forget” their original directions. Therefore, the more the walls quiver due to thermal agitation, the better their performance.

As the simplest example consider a long thin tube filled with a rarefied gas. Horodeński⁽⁵¹⁾ concludes that more molecules will fly parallel to the tube than across it. This seems plausible, since molecules moving parallel to the tube have longer free paths. Therefore, if particles are allowed to migrate between the end of one tube and middle of another, as in Fig. 3, there should be a net migration from end to middle. It follows that a clockwise wind is expected in Fig. 3, which could operate a turbine.

Another example is sketched in Fig. 4. Two chambers are connected through a microscopic orifice, located at the extremity of a funnel-like surface. This surface guides the molecules in the lower chamber to the orifice. On the other hand, molecules in the upper chamber have a minute probability of escaping through the orifice. From this asymmetry we could expect a pressure difference to build up, which could be used to extract work. Devices of this kind are successfully used as traps for flies.

In the following examples an intrinsic asymmetry will be introduced to the motion of the particles. This may be achieved by means of magnetic or Coriolis forces, which bring about a difference between turning to the left or

to the right.

The first case is sketched in Fig. 5a. The ring represents a piece of metal in which the free electrons have very long free paths, so that they collide only against the boundaries of the ring. These electrons have thermal motion; in the absence of electromagnetic fields, they move along straight trajectories until they hit a boundary of the ring. At the boundary, the electron velocity is randomized (in accordance with the temperature of the ring) and a new straight trajectory begins. We classify the trajectories into two groups: trajectories in group I contribute to a clockwise current around the ring and trajectories in group II contribute to a counterclockwise current (currents of electrons; the negative charge of the electron is not essential for the present argument). On the average, both currents are equal and the net current is zero.

Let us now consider the case in which the ring is embedded in a magnetic field, perpendicular to the plane of the ring, such that the electrons move in counterclockwise circular orbits. In this case trajectories in group I will be curved opposite to the ring and will hit the boundary sooner than in the zero-field case. Contrariwise, trajectories in group II bend in the same sense as the boundary and will therefore have lengthened lives. As a result, trajectories in group I are inhibited, in group II are enhanced and a non-zero net current (which can run a motor) should be expected. While imparting energy to the motor, electrons slow down. Their energy is not replenished by the static magnetic field, since the work performed by the magnetic force is zero. Rather, they pump thermal energy from the environment (when they rebound at the walls), since its temperature remains unchanged.

But Nature is stubborn. There are some trajectories, such as that shown (enlarged) in Fig. 5b, which contribute to a current in “the wrong” sense when the magnetic field is present. If the velocity distribution of the rebounding electrons were such that there are many trajectories as in Fig. 5b and very few grazing trajectories, then the entire effect could cancel out. This is indeed

the case.^(53,54) The distribution of the rebounding particles is precisely the one which is required in order that no net current builds up in Fig. 5, no wind builds up in Fig. 3 and no pressure difference builds up in Fig. 4. The latter device works for flies, because they have prejudices, but molecules don't (as far as we know).

We shall close this subsection with a more sophisticated situation. Let us consider a gas of polarizable particles (as most particles are). We assume that the particles can be excited by some agent, say electromagnetic radiation. Let there be an electric field in the y -direction and a magnetic field in the z -direction.

When a particle becomes excited, the interaction between its positive and negative components becomes weaker and therefore its polarization increases. This means that the positive components will move in the y -direction relative to the negative components, and the particle will experience a net magnetic force in the x -direction. As a result, when a particle is excited it will be granted some momentum Δp in the x -direction, which is proportional to the magnetic field and to the change in polarization. When the particle is de-excited, the change in momentum is $-\Delta p$. We might therefore expect a flow of excited particles in the x -direction and a flow of de-excited particles in the opposite direction, namely, heat flow in the x -direction.⁽⁵⁵⁾

Preliminary simulations which I once performed did not provide evidence for the expected heat flow. Nevertheless, it seems to me that this situation deserves further study.

Related ideas have been proposed by several authors.⁽⁵⁶⁾

5. Typical and Exceptional Measurements

As stated in Subsection 3.2, the entropy of the system may be identified with the lack of information we have about the state of the system. The word “we” may disturb those who are used to think of the entropy as a function of the system only; “we” are those who decide which are the macroscopic

coordinates, measure them and control them. “Our” role has been discussed in Section 3. As a consequence, if we increase our knowledge by means of a measurement, then the entropy of the system should decrease, even if the microstate of the system remains unchanged. This possibility is what Smoluchowski had in mind when he suspected that “intelligent beings” might overcome the second law.

The old answer to this apparent violation of the second law is that the entropy of the measuring instrument must increase by at least as much as the entropy of the system decreases. The question of whether it is or is not essential to invest useful energy in order to perform a measurement has been the subject of extensive discussion.^(13,28,51,57–60)

For the purpose of the present section, we may disregard the question of whether measurements could in principle reduce the entropy and consider only the quantitative aspect of this process. How much could a measurement help? For instance, if we are dealing with a system of n particles and in the ideal event that we exactly measure the state of one particle, how much information do we gain? The remaining lack of information would be that of the remaining $(n - 1)$ molecules. Therefore, if n is comparable to Avogadro’s number, the decrease in entropy would have no practical consequence. More generally, what measurements do is discard possibilities. For example, if before the measurement there were N conceivable states ψ_i , and the measurement adds three significant digits to the previously known value of some quantity, then, after the measurement there would typically remain $N/1000$ possible states ψ_j which could for instance be those with $j = 1000, 2000, 3000, \dots N$. Let’s denote by H' the value of H after the measurement. $H' = -\sum P'_j \ln P'_j$, with P'_j the probability for the system being in the state ψ_j , as inferred from the measurement. For simplicity, let’s assume that if the system were known to be before the measurement in any of the states ψ_i in the interval $j - 999 \leq i \leq j$, then it would be known to be in ψ_j

after the measurement. This means

$$P'_j = \sum_{i=j-999}^j P_i . \quad (23)$$

It follows that the increase in H due to the measurement is

$$\begin{aligned} \Delta H = H' - H &= \sum_j \sum_{i=j-999}^j P_i \ln P'_j - \sum_j \sum_{i=j-999}^j P_i \ln P_i \\ &= \sum_j \sum_{i=j-999}^j P_i \ln(P'_j/P_i) . \end{aligned} \quad (24)$$

This increase attains its maximal value when all the probabilities in every group $j - 999 \leq i \leq j$ were equal before the measurement, as would be the case if all the states in the group belonged to the same class $G_{\nu(j)}$ of previously indistinguishable states. In this case $P_i = P'_j/1000$, and we achieve an entropy decrease $-\Delta S = k\Delta H_{\max} = k \ln 1000$. Had the measurement provided ten significant figures, then we would achieve $-\Delta S = 10k \ln 10$, which is still a microscopic quantity.

Since only macroscopic amounts of entropy are thermodynamically significant, a macroscopic number of measurements is needed to have any influence in this context. We might still hope to perform all these measurements, were it not for three strong limitations: (a) they have to be reversible, otherwise useful energy would be dissipated; (b) they have to be concluded before the state of the system changes and they become irrelevant; and (c) they have to measure independent quantities. These limitations rule out the possibility of long sequences of measurements; it is necessary to perform macroscopic numbers of measurements simultaneously. But reversible measurements involve^(28,58,59) gradual motion of mechanical components or gradual variation of fields, as well as permanent equilibration with thermal reservoirs. In summary, the decrease of entropy by means of typical measurements requires control over a macroscopic number of macroscopic devices. Such a

monster seems to be a subject for philosophical preoccupation rather than a physically significant menace to the second law.

The quantitative performance of a hypothetical Maxwellian demon has also been considered by Leff.⁽⁶¹⁾ From his results it can be concluded that a single demon would have practically no effect; only a team of demons might have significant influence.

We still have to consider the case of “exceptional” measurements. For example, let us consider the situation proposed by Gabor.⁽⁶²⁾⁴ Fig. 6 shows a long cylinder of volume V with a single molecule in its interior. Every time the molecule hits the walls of the cylinder, it rebounds with a new velocity. These velocities cannot be predicted deterministically, but they comply with the Maxwell-Boltzmann distribution corresponding to the temperature of the cylinder (which is the same as the temperature T of the reservoir). Using kinetic theory, it is easy to show that the average force exerted by the molecule on a unit surface multiplied by the volume available to the molecule equals kT , so that the average work performed by the molecule when moving a piston will be a particular case of an ideal gas which changes its volume.

We assume that the molecule has some detectable feature, such as a magnetic moment, which enables detection of the presence of the molecule within a volume v . We could then place a detector at an end of the cylinder and wait until the molecule is found within the volume v close to that end. This situation may be properly likened to a spider waiting for its prey. Once the molecule is detected, it is imprisoned by the insertable piston P . Enviewing the single molecule as an ideal gas which has changed its volume from V to v , its entropy will have decreased by the amount $-\Delta S = k \ln(V/v)$. With the help of a mechanical device W , our single-molecule gas is now allowed to expand, while performing work on W . For a quasistatic, frictionless and isothermal expansion, the amount of performed work will be $kT \ln(V/v)$, with T the prevailing temperature. In expanding, the entropy of the gas returns to its previous value; but this time the entropy of the heat reservoir decreases

by $k \ln(V/v)$, violating the second law. By choosing V sufficiently large, the gain in useful energy should more than compensate for any possible losses in the operation of this engine.

However, increasing the volume V is a gamble. If the detector D is at the non-zero temperature T , it will sometimes be fooled by thermal fluctuations and wrongly believe that the molecule is there. Therefore, there is always a risk of undergoing too many false alarms, and exhausting the useful energy in repairing their consequences, before the molecule is caught. If we choose a huge value for V , then we should expect to wait a very long time until the molecule happens to arrive and this risk becomes certain failure.

As a closer example to real life, we consider a case in which a physicist receives two barrels of water. Both barrels look the same and the physicist might pour them together into a bigger container, without suspecting that this process has any significant influence on the entropy of the water. Let's assume, however, that the barrels were brought in an open truck; one of them stood under the sun most of the journey, and the other one was in the shade. Therefore, if the physicist happens to measure their temperatures, he or she will discover that they are not equal. This discovery enables him or her to extract a macroscopic amount of useful energy.

At this point it is essential to notice that the hard stage in extracting the above useful energy was not in measuring the temperatures, but rather in guessing that there might be a temperature difference. It could just as well be the case that both barrels were at the same temperature but, say, the salt concentrations were different. Translated into the situation of Fig. 6, an exceptionally good guess would be to locate the detector in the right place, so that the molecule arrives in a short time in spite of the ratio V/v being huge. Measurements like these, which by conservative thinking are expected to produce a void answer, but in exceptional situations could provide a macroscopic amount of information, are what I call “exceptional measurements”.

Finally, let me express exceptional measurements in the language of algo-

rithmic complexity.^(25–29) Given a long string of digits, an exceptional measurement consists of finding (actually, guessing) two simple algorithms which achieve the following: the first assigns a weight to the position of every digit and the second predicts the values of the digits with a success rate which, properly weighted by the first algorithm, is appreciably larger than expected for random digits. The first algorithm defines the macroscopic coordinate which is being measured, and the second expresses compressibility. I am not aware of any mathematical study of the size of the feasibility domain for this process.

6. Interpretation

The rebel’s inquiry can be made to comply with the second law if we accept the view that “order” is not a property of a system, but rather a property of the observer describing it.

6.1 Obvious order and encoded order

The universe is always in some definite state, which we shall denote by ϕ . (The meaning of universe is an assembly of objects which interact only among themselves.) Usually, ϕ is a complicated linear combination of the states ψ_i , which were chosen according to the best of the observer’s knowledge.

The “objective probability” of having the universe in its state ϕ is always 1, and the “objective probability” for any other orthogonal state is always 0. Therefore, according to definition (5), the “objective entropy” of the universe is always zero.

Roughly speaking, we say that the state of a system is ordered when we can recognize in it some intelligible pattern and we can predict the evolution of the state with some microscopic detail. A state has “encoded order” if order can be found in it by looking at the correct combination of coordinates. Mathematically, encoded order can be transformed into order by a suitable mapping. Let us consider a few examples as an illustration of what I mean

by “encoded order”.

a. Let us consider our solar system and an observer from outer space which looks at us once every several years. If there were no gravitation and motion, the observer would always see us at the same positions and say that our state is ordered. But since there is gravitation and motion, the positions of the planets at each observation will apparently be unrelated to the previously observed positions and the first conclusion should be that the solar system is undergoing chaotic motion. However, if the observer knows Kepler’s laws, then the observed positions will fit into orbits which describe the itinerary of the planets before, between and after observations, and the order will be as perfect as in the case of motionless planets.

b. Any computer language deals with “integers” $1, 2, 3, \dots$, which have a natural order. A normal person knows which is the succeeding integer in the sequence. This is plain order. Most computer languages can also generate “random numbers”, such as 6835258, 9110669, 7652114, ... This sequence is actually a permutation of integers, designed to avoid any correlation between the values of the numbers and their order of appearance. Namely – complete disorder. However, someone who knows the algorithm and the seed from which the random numbers are generated, can write a program which assigns the proper ordinal to any value, or vice-versa. Namely – complete determinism. For the algorithms used by most computer languages, the length of this program would be quite small, and has nothing to do with the number of permutations which can be performed among the integers. (Discovering the algorithm would be a different story.)

c. Consider l ideal pendula ($l \gg 1$), with periods $\tau/(p+1)$, $\tau/(p+2)$, ... $\tau/(p+l)$, where p is some integer. All the pendula are launched together, initially in phase. After some time of the order of τ/l , the pendula are completely out of phase. If the location of every pendulum is unrelated to its period, their motion looks chaotic. However, after time τ , every pendulum will have undergone an integer number of periods, and they will all

be in phase. One might say that order was destroyed shortly after launching and created again shortly before time τ ; however, if we number the pendula by their period lengths and write down their equations of motion, it becomes more sensible to assert that order is always present: it is entangled most of the time and shows up every time τ .

d. Consider a sample with all its spins initially in the same direction. This alignment may be thought of as order. If the spins are perpendicular to the magnetic field, they precess around the field axis. Due to the inhomogeneity of the sample, each spin precesses at a slightly different angular velocity and at a time t_1 the angular distribution of the spins will have become isotropic in the precession plane. This isotropic distribution may be thought of as disorder. With an appropriate technique, the orientations of all the spins at time t_1 can be inverted. After this inversion, the orientation of every spin is the same as if it had been precessing backwards. Therefore, at time $2t_1$, every spin will simultaneously be back at its original orientation and the sample will have returned to the original ordered situation. We may therefore say that at time t_1 there was encoded order and that during the interval $t_1 \leq t \leq 2t_1$ this order was decoded. The effect described here is well known and measurable; it is called “spin echo”.

e. When light from an object passes a piece of glass, its phase is shifted. If the glass is frosted, this phase-shift is a complicated function of position, and the image of the object becomes unrecognizable, as in Fig. 7a. With an appropriate technique, these phase-shifts can be reversed and the original image is recovered, as shown in Fig 7b.⁽⁶³⁾

Although we can always imagine a mapping which transforms encoded order into readily observable order, it should not be understood from the examples above that perfect decoding is physically attainable. As in the case of Maxwell’s demons, decoders are subject to thermal vibrations which limit the amount of order which can be recovered. Thus, while a mathematical computer is assumed to be capable of sorting any list of numbers, a physical

computer is expected to introduce mistakes if the number of elements in the list is of the order of Avogadro's number; likewise, while the technique in example (e) can restore features of optical wavelength sizes, it cannot cope with features of atomic size. For an initially ordered state it is customary to define two different relaxation times: a time τ_1 after which the initial order apparently fades away, and a time τ_2 after which the initial order cannot be recovered by means of the available decoding technique.

The concept of encoded order may seem to be incompatible with the concept of algorithmic randomness.^(25–29) Even if there exists a mapping which transforms a complicated state ϕ into a simple state, say ψ_{100} , the mapping may be as complicated as ϕ , so that handling it is as impractical as handling ϕ . We shall return to this point in the following subsection.

6.2 The source of useful energy

Thermodynamics relates the useful energy to the entropy, and Section 3 expresses the entropy as a function of the statistical state $\{P_i\}$. According to Section 3, the probabilities $\{P_i\}$ don't describe the objective evolution of the system, but rather the evolution of the knowledge which the observer has about the system. Obviously, an entity which doesn't feel the energy around it and cannot decide what to do with that energy, has no useful energy at its disposal.

The lesson called for is that the useful energy available from a system depends on the information available about the state of the system. The meaning of Boltzmann's theorem is that, since the observer is unable to follow in detail the evolution of the universe, his or her knowledge about the universe (and therefore his/her ability to use its energy) will decrease as time elapses. The second law does not imply that order turns into "disorder". What happens is that readily observable order transforms into encoded order. If we knew how to perform the appropriate decoding, entropy would remain constant.

But how can we always lose something which we don't have? Before we can lose information about our universe, we necessarily have to gain it. We might expect to gain information by means of measurements, but Section 3 and all the literature on “intelligent demons”^(13,50,51,57–60) indicate that “typical measurements” won't help. An increase in our knowledge about the universe will occur if and only if we are lucky enough to perform an “exceptional measurement”, as described in the previous section.

If we lived in a random universe, it would be utterly hopeless to trust on “exceptional measurements” as our initial source of useful energy. But the universe is not random; on the contrary, it is always in some definite state ϕ . For any given state ϕ , the universe favours those creatures which are sensitive to some coordinates that give rise to “exceptional measurements”. For example, when we decide whether to eat an apple or not, the number of parameters which we really check is not very large; it could be roughly estimated by dividing the cross section of an optic nerve by the cross section of an axon. In order to decide with certainty whether that apple is good or dangerous to our health, the number of parameters which we ought to check is at least of the order of the number of nucleotides of our genome, which is greater by roughly three orders of magnitude. It is therefore a miracle that we usually succeed. The only reason that nearly everything that looks and tastes like an apple is indeed an apple, is that we live in a non-random environment, and our senses are adapted to filter non-random coordinates.

In “exceptional measurements”, use is made not only of the information provided by the measurement, but mainly of previous knowledge about the prevailing order in the environment. When we detect an oil field, the amount of useful energy which we can control is proportional to the size of the field, and is practically unrelated to Eq. (24).

Let us now confront the problem of algorithmic randomness. The problem is best stated in Ref. 29. Let us assume that at some moment, say at big-bang time, the universe was in a simple state ϕ_0 . After some time Δt , the universe

will be in a not too different state, $\phi_1 = \mathcal{U}\phi_0$. In general, after time $\ell\Delta t$, the state of the universe will be $\phi_\ell = \mathcal{U}^\ell\phi_0$. Only after a very long time $L\Delta t$, which is the Poincaré time, the state of the universe will be $\phi_L \approx \Phi_0$ and will again become simple. For a general state ϕ_ℓ it will be true that it could be transformed, in principle, into ϕ_0 , but in order to specify the required transformation we should stipulate the very long number ℓ . Handling the number ℓ brings about a thermodynamic burden which for most cases is effectively equivalent to an additional amount of entropy $\sim k \ln \ell \sim k \ln L$.

I claim that the limitation referred to in the previous paragraph is overcome by successful observers. The entire formalism of algorithmic randomness is valid for “universal computers”,^(25–27) namely, the length of the language which translates between two computers is negligible in comparison with the length of the strings which they handle. Successful observers are not universal computers; they have evolved hand in hand with their environment and have some number ℓ_0 , such that $|\ell - \ell_0| \ll \ell$, built in within their “hardware”. Therefore, successful observers have to find relatively simple transformations, like $\mathcal{U}^{\ell-\ell_0}$, and, accordingly, the environment does not look chaotic to them.

When useful energy is dissipated, the order of the universe changes, and new coordinates become non-random. As the order of the universe changes, new creatures, which are sensitive to these new coordinates, are more suited to succeed. The rebel’s conclusion could rightly be “I have useful energy (which enables me to think), therefore I must have been seeing order in my environment”.

6.3 Revision of thermodynamic concepts.

From Eq. (1) it follows that

$$\frac{\partial S}{\partial U} = \frac{1}{T}; \quad \frac{\partial S}{\partial V} = \frac{p}{T}, \quad (25)$$

where U, V, T and p stand for the internal energy, the volume, the tempera-

ture and the pressure, respectively.

I have been claiming that S depends on the complete set of coordinates chosen by the observer. But, can the temperature or the pressure, which are directly measured, depend upon this choice? By definition, they could. Eq. (25) involves partial derivatives, and their values depend on which arguments are kept constant in taking the derivative. When going from (1) to (25) it has been assumed that these arguments are, besides U or V , all the macroscopic coordinates in the complete set such that varying them would mean doing work.

In order to have a precise idea of the significance of Eq. (25) to the measurement of T and p , let us first deal with an artificially simple model. The conclusions will then be generalized.

We consider a system whose state is described by f indices i_1, i_2, \dots, i_f , where each index can take on the values $i_j = 0, 1, 2, \dots$. The energy E of the system is postulated to be

$$E = \epsilon \sum_{j=1}^f i_j , \quad (26)$$

where ϵ is a constant. This model is known as the Einstein model for f modes of excitation.

Let E be the only macroscopic coordinate which is measured by the observer A . For simplicity, we shall assume that A can measure E exactly. For a given value of E , the number $\Omega_f(E)$ of possible microstates is determined by the recursion relation

$$\Omega_l(E) = \sum_{i=0}^{E/\epsilon} \Omega_{l-1}(i\epsilon); \quad \Omega_1(E) = 1 . \quad (27)$$

For $E \gg \epsilon$ (“classical” limit), (27) yields

$$\Omega_f(E) = \frac{1}{(f-1)!} \left(\frac{E}{\epsilon} \right)^{f-1} . \quad (28)$$

Since the value of E is the only information A has about the state of the system, the probability for each microstate is just $1/\Omega_f(E)$ and the entropy is

$$S_A(E) = k(f-1) \ln(E/\epsilon) - k \ln(f-1)! \quad (29)$$

Therefore, the temperature T_A is given by

$$1/T_A = dS_A/dE = k(f-1)/E . \quad (30)$$

Let observer B measure, in addition to E , the quantity

$$D = \epsilon \sum_{j=1}^{f_D} i_j , \quad (31)$$

with $0 < f_D < f$. We have chosen to consider this quantity, since it is neither independent of nor a function of E . For given values of E and D , the number of possible microstates will be $\Omega_{f_D}(D)\Omega_{f-f_D}(E-D)$. Since there is no reason to associate to any of these states a preferred probability over the others, the entropy is

$$\begin{aligned} S_B(E, D) &= k(f - f_D - 1) \ln[(E - D)/\epsilon] + k(f_D - 1) \ln(D/\epsilon) \\ &\quad - k \ln[(f - f_D - 1)!(f_D - 1)!] \end{aligned} \quad (32)$$

and the temperature T_B is given by

$$1/T_B = (\partial S_B / \partial E)_D = k(f - f_D - 1)/(E - D) . \quad (33)$$

In general, $T_B \neq T_A$. Note that if D maximizes $S_B(E, D)$ and $f \gg 1$, then $T_B \rightarrow T_A$, the reason being that the knowledge about D is just the same which could have been anticipated from the knowledge of E . In this case, B does not have more knowledge than A .

For observer B it would be more natural to regard the system as being composite: part of it comprises the first f_D excitation modes, has energy $E_1 =$

D , entropy $k(f_D - 1) \ln(E_1/\epsilon(f_D - 1)!)$ and temperature $T_1 = E_1/k(f_D - 1)$; the second part comprises the remaining $f - f_D$ modes, has energy $E_2 = E - D$, entropy $k(f - f_D - 1) \ln(E_2/\epsilon(f - f_D - 1)!)$ and temperature $T_2 = E_2/k(f - f_D - 1) = T_B$.

But how does the thermometer know whether to measure T_A , T_1 or T_2 ? What the thermometer measures is its own energy, reached in equilibrium with the system, under well specified thermal contact. A thermometer designed to measure T_1 (resp. T_2) should exchange energy with the first f_D (resp. the last $f - f_D$) excitation modes of the system, but not with the remaining modes. A thermometer designed to measure T_A should exchange energy with all the excitation modes. Actually, what thermometer A measures is the intermediate temperature

$$T_{meas} = cT_1 + (1 - c)T_2 \quad (34)$$

where c depends on the relative strength with which the thermometer is coupled to the excitation modes in both sets. If the average coupling per mode is the same for the first f_D and for the last $f - f_D$ excitation modes, and considering $f, f_D \gg 1$, then $c = f_D/f$ and $T_{meas} = T_A$. If neither $T_1 = T_2$, nor is the coupling strength independent of the set to which an excitation mode belongs, then $T_{meas} \neq T_A$, and different thermometers could give different results. This could be a clue for observer A , indicating that an important feature about the state of the system is being overlooked.

If E_1 and E_2 differ by at most a few orders of magnitude, whereas f_D is smaller than f by *many* orders of magnitude, then T_1 will effectively become infinity. From the practical point of view, an apparatus which exchanges energy only with the first f_D modes will not be called a thermometer or a heat bath, but rather a resonator. T_1 being infinity, the Carnot efficiency when using E_1 to perform work would be 100%. In this case, E_1 is called mechanical energy, E_2 is called the internal energy and $T_2 = (\partial E_2 / \partial S_B)_{E_1}$ is called the temperature of the system.

Let us now generalize the conclusions obtained from this model. We regard Eq. (25) as the definition of temperature. In order to fit this definition, thermometers should be made so that they do exchange internal energy but are impermeable to all the remaining macroscopic coordinates in the complete set. For instance, let's consider atmospheric temperature. An observer who is aware of evaporation, will always be careful to distinguish between the temperatures read by a dry or by a wet thermometer. On the other hand, an observer who doesn't suspect that vapors exist, will at best notice that different thermometers show different temperatures due to some unknown reason, and will be forced to conclude that there is a transient irreproducible behaviour unless saturation is reached.

For typical situations, the observer is able to classify the microscopic degrees of freedom of the system into several groups, such that the interaction among different groups is weak. The possible criteria for this classification are unlimited: location, chemical composition, kind of motion, interaction with static fields, interaction with radiation, resonance frequency, etc., and all possible combinations. Accordingly, an observer A will define a groups, each with its own complete set of macroscopic coordinates, including internal energy U_α , entropy S_α and temperature

$$T_\alpha = \frac{\partial U_\alpha}{\partial S_\alpha} \quad \alpha = 1, 2, \dots a . \quad (35)$$

Similarly, an observer B could define a different set of groups and a different set of temperatures

$$T_\beta = \frac{\partial U_\beta}{\partial S_\beta} \quad \beta = 1, 2, \dots b . \quad (36)$$

All that has been said for temperatures can be translated to the case of pressures. The force per unit area exerted on an instrument depends not only on the state of the system, but also on the instrument. The higher the permeability of the walls of the barometer to a kind of particles, the lower

the pressure which these particles will exert on them. If the properties of the different components of a mixture are known, then barometer walls can be designed so that they are either completely permeable or completely impermeable to appropriate groups of components; in this way different barometers add independent pieces of information. There is no essential difference between “partial pressures” and the temperatures T_α of Eqs. (35)-(36), except for the experimental fact that it is very easy to vary the volume of all the groups by the same amount, while it is very difficult to vary their energies by the same amount.

It is the ambiguity in the classification of the degrees of freedom of the system, as manifested by Eqs. (35) and (36), which makes it possible to bring the rebel and Lord Kelvin to terms. According to Kelvin’s formulation of the second law, if a closed system *is* at uniform temperature (and all the other intensive parameters are uniform too), then *it is* impossible to transform part of its internal energy into mechanical energy. The present interpretation reformulates the second law in the form: “If *an observer is unable to discover* in a closed system *any classification* such that at least two of the temperatures (or any other intensive parameter) in his/her set of groups are different, then *this observer* is unable to transform part of the internal energy of the system into mechanical energy.”

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References

1. J.A. Wheeler and W.H. Zurek, eds. *Quantum Theory and Measurement* (Princeton Univ., N.J., 1983).
2. J.S. Bell, *Speakable and unspeakable in quantum mechanics* (Cambridge University Press, Cambridge, 1987).
3. K.G. Denbigh and J.S. Denbigh, *Entropy in Relation to Incomplete Knowledge* (Cambridge University Press, Cambridge, 1985).
4. W.H. Zurek, ed., *Complexity, Entropy and the Physics of Information* (Addison-Wesley, Redwood, U.S.A., 1990).
5. L. Boltzmann, Sitzungsberichte Akad. Wiss. (Vienna) part II, **66**, 275 (1872), translated in Ref. 6.
6. S.G. Brush, *Kinetic Theory* (Pergamon, Oxford, Vol. 1 - 1965, Vol. 2 - 1966, Vol. 3 - 1972).
7. P. Ehrenfest and T. Ehrenfest, *The Conceptual Foundations of the Statistical Approach*, English Translation (Cornell University, Ithaca, 1959).
8. R.C. Tolman, *The Principles of Statistical Mechanics* (Dover, N.Y., 1979).
9. P.C.W. Davies, *The Physics of Time Asymmetry*, (University of California, 1974).
10. M.C. Mackey, Rev. Mod. Phys. **61**, 981 (1989).
11. W. Shannon and W. Weaver, *The Mathematical Theory of Communication* (Univ. of Illinois, 1949).

12. A.I. Khintchin, *Information Theory* (Dover, N.Y., 1957).
13. L. Brillouin, *Science and Information Theory*, 2nd ed. (Academic, N.Y., 1962).
14. R.B. Ash, *Information Theory* (Wiley, N.Y., 1965).
15. R.W. Hamming, *Coding and Information Theory* (Prentice Hall, N.J., 1986).
16. E.T. Jaynes, Phys. Rev. **106**, 620 (1957); **108**, 171 (1957).
17. E.T. Jaynes in *The Maximum Entropy Formalism*, R.D. Levine and M. Tribus, eds. (MIT Press, Cambridge Mass., 1978).
18. E.T. Jaynes, *Papers on probability, statistics and statistical physics*, Synthese Library, Vol. 158 (R.D. Rosenkrantz, ed.) (Reidel, Dordrecht, 1983).
19. E.T. Jaynes, in *Complex Systems – Operational Approaches*, H. Haken, ed. (Springer-Verlag, Berlin, 1985).
20. A. Katz, *Principles of Statistical Mechanics* (Freeman, San Francisco, 1967).
21. A. Hobson, *Concepts in Statistical Mechanics* (Gordon and Breach, N.Y., 1971).
22. R. Balian, Y. Alhassid and H. Reinhardt, Phys. Rep. **131**, 1 (1986).
23. J.C. Maxwell, “Diffusion”, *Encyclopedia Britannica*, 9th ed. (N.Y., 1878), vol. 7, 220.
24. Course in Analytic Inventive Thinking, The Open University of Israel, unpublished. A more accessible related reference is G.S. Altshuller, *Creativity as an Exact Science* (Gordon and Breach, N.Y., 1984).

25. R.J. Solomonoff, Inf. Control **7**, 1 (1964).
26. A.N. Kolmogorov, Inf. Transmission **1**, 3 (1965).
27. G.J. Chaitin, J. Assoc. Comput. Mach. **13**, 547 (1966).
28. C.H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982).
29. W.H. Zurek, Phys. Rev. A **40**, 4731 (1989), and references therein.
30. J. Berger in *Horizons of Physics*, Vol. 2, N. Nath, ed., series editor A.W. Joshi (Wiley Eastern, New Delhi, 1994?).
31. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ., 1955). Relevant chapters reproduced in Ref. 1.
32. S.W. Hawking, Phys. Rev. D **13**, 191 (1976).
33. S. Frautschi, Science **217**, 593 (1982).
34. P.T. Landsberg, *The Enigma of Time* (Adam Hilger, Bristol, 1982).
35. O. Penrose, *Foundations of Statistical Mechanics*, (Pergamon, Oxford, 1970).
36. O. Klein, Zeits. Phys. **72**, 767 (1931).
37. E.C.G. Stueckelberg, Helv. Phys. Acta **25**, 577 (1952).
38. E.T. Jaynes, Am. J. Phys. **33**, 391 (1965).
39. E.T. Jaynes, Phys. Rev. A. **4**, 747 (1971).
40. J. Berger, Eur. J. Phys. **11**, 155 (1990).
41. W. Pauli in *Sommerfeld Festschrift*, P. Debye, ed. (Hirzel, Leipzig, 1928).

42. P.T. Landsberg, Phys. Rev. **96**, 1420 (1954).
43. J. Hurley, Am. J. Phys. **54**, 25 (1984).
44. R. Peierls, *Surprises in Theoretical Physics* (Princeton Univ., N.J., 1979).
45. M. Smoluchowski, Phys. Z. **13**, 1069 (1912); M. Smoluchowski in *Vor-trage uber die Kinetische Theorie der Materie und der Elektrizitat*, M. Planck, ed. (Teubner, Leipzig, 1914).
46. R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1963), Vol. 1, ch. 46.
47. L.G.M. Gordon, Found. Phys. **11**, 103 (1981) and **13**, 989 (1983).
48. L. Brillouin, Phys. Rev. **78**, 627 (1950); R. McFee, Am. J. Phys. **39**, 814 (1971); R.E. Burgess, ed., *Fluctuation Phenomena in Solids* (Academic, N.Y., 1965); J.C. Yater, Phys. Rev. A **10**, 1361 (1974); E.P. EerNisse, Phys. Rev. A **18**, 767 (1978).
49. P.A. Skordos and W.H. Zurek, Am. J. Phys. **60**, 876 (1992).
50. H.S. Leff and A.F. Rex, Am. J. Phys. **58**, 201 (1990).
51. H.S. Leff and A.F. Rex, *Maxwell's Demon: Entropy, Information, Computing* (Princeton Univ. and Adam Hilger, 1990).
52. A. Horodeński, J. Vac. Sci. Technol. A **3**, 39 (1985); Phys. Lett. A **122**, 295 (1987); J. Berger, J. Vac. Sci. Technol. A **5**, 382 (1987).
53. J. Berger, Am. J. Phys. **53**, 899 (1985).
54. H.P. Steinrück, K.D. Rendulic and A. Winkler, Surface Science **154**, 99 (1985).

- 55. J. Berger, Solid State Com. **53**, 387 (1985).
- 56. H. Aspden, Nature **347**, 24 (1990).
- 57. L. Szilard, Z. Phys. **53**, 840 (1929), translated in Ref. 1.
- 58. C.H. Bennett, Scientific American **257**, 88 (November 1987).
- 59. J. Berger, Int. J. Theor. Phys. **29**, 985 (1990).
- 60. C.M. Caves, W.G. Unruh, and W.H. Zurek, Phys. Rev. Lett. **65**, 1387 (1990).
- 61. H.S. Leff, Am. J. Phys. **58**, 135 (1990).
- 62. D. Gabor, Progress in Optics **1**, 111 (1964), reprinted in Ref. 50.
- 63. J. Feinberg, Optics Letters **7**, 486 (1982).

Endnotes

1. This idiom has been taken from Ref. 2.
2. In case of doubt about what is meant by inference or probabilities, let me state that their meanings are the same as in Ref. 21.
3. It is known, though, that Eq. (2) is insufficient to predict macroscopic evolution when the system undergoes an instability. For instance, if at time t_o a ferromagnetic sphere slightly above its Curie temperature is immersed in a cool fluid, then it is experimentally found that some time later the sphere becomes magnetized in a definite direction. However, if at time t_o there is perfect spherical symmetry (to the best of our knowledge about the material and about the present fields), then the evolution rule (2) is unable to break that symmetry.
4. Landauer and Bennett have shown that the original analysis of this problem was inadequate. See, for instance, R. Landauer in *Selected Topics in Signal Processing*, S. Haykin, ed. (Prentice-Hall, Englewood Cliffs, N.J., 1989).

Figure Captions

Fig. 1: Two compartments separated by a door which opens when kicked from the right, but not when kicked from the left. The spring closes the door after the molecule M has passed through.

Fig. 2: A closed circuit, composed of a resistor, a diode and a motor.

Fig. 3: Four long thin tubes, such that an orifice connects between an end of each and the middle of its neighbour. If more molecules are moving parallel to the tubes than across them, then there will be wind $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$.

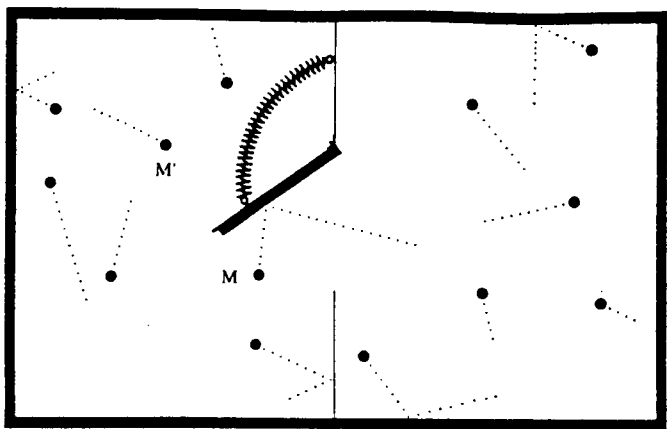
Fig. 4: Two compartments separated by a funnel-shaped surface, with a microscopic orifice in the middle. The surface is intended to “help” the molecules reach the orifice when coming from below.

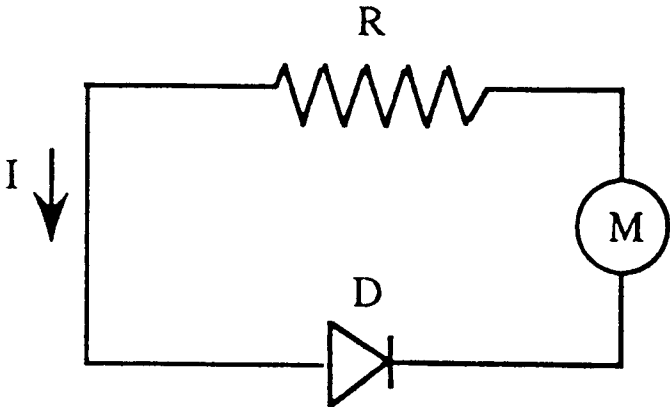
Fig. 5: Orbits described by free electrons inside a conducting ring. Their trajectories are bent by a magnetic field perpendicular to this page. (a) I and II depict two possible orbits. Presumably, these are the typical cases. (b) Another possible orbit, which must be taken into account.

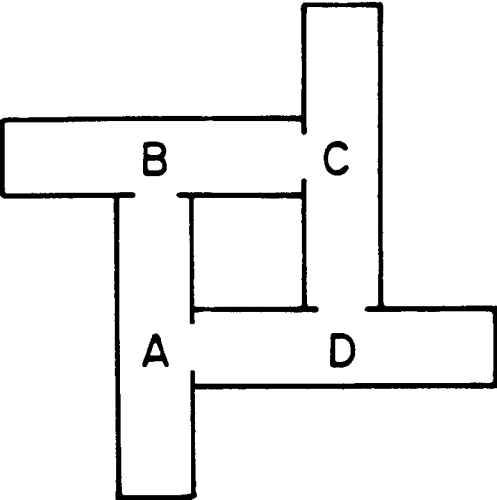
Fig. 6: A very long cylinder of volume V , with a small region v where the molecule can be detected and trapped. M: molecule; D: detector; P: insertable piston; T: heat reservoir; W: mechanical device, tailored to respond to pressure on the piston and deliver mechanical energy.

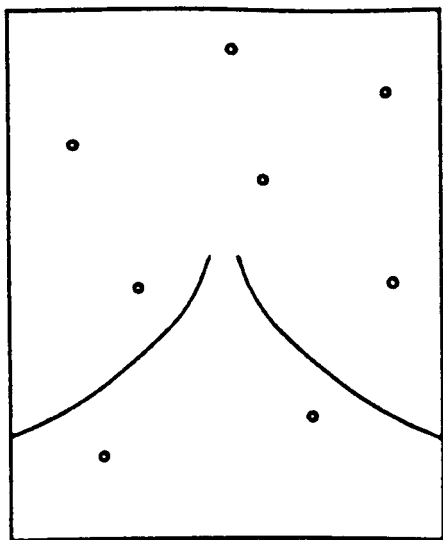
Fig. 7a: Image of a recognizable object, distorted by a frosted glass.

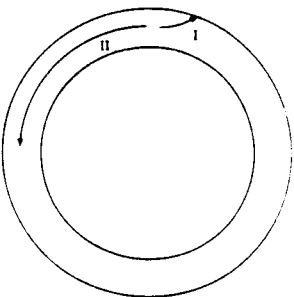
Fig. 7b: The image of Fig. 7a, restored by means of the technique described in Ref. 63.



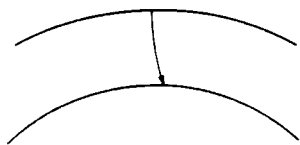








(a)



(b)

